Stability analysis of BHs by the S-deformation method

MK, CQG **34**, 235007 (2017)
MK & T.Tanaka, CQG **35**, 195008 (2018)
MK & T.Tanaka, arXiv:1809.00795

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Linear (mode) stability of BH

Linear gravitational perturbation on a highly symmetric BH usually reduces to

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}-V(x)\right]\tilde{\Phi}=0$$

$$\tilde{\Phi}(t,x)=e^{-i\omega t}\Phi(x)$$

$$\left[-\frac{d^2}{dx^2}+V\right]\Phi=\omega^2\Phi$$

unstable mode → $\omega^2<0$ mode

(negative energy bound state)

To prove (mode) stability, we need to show the non-existence of $\omega^2 < 0$ mode

$$\left[-\frac{d^2}{dx^2} + V\right]\Phi = \omega^2 \Phi$$

$$\Longrightarrow \left[\bar{\Phi}\frac{d\Phi}{dx}\right]_{-\infty}^{\infty} + \int dx \left[\left|\frac{d\Phi}{dx}\right|^2 + V|\Phi|^2\right] = \omega^2 \int dx |\Phi|^2$$

$V \geq 0$ implies non-existence of $\omega^2 < 0$ mode

Sometimes, V contains negative regions

S-deformation **[Kodama and Ishibashi 2003]**

$$-\frac{d}{dx}\left[\bar{\Phi}\frac{d\Phi}{dx}+S|\Phi|^2\right]+\left|\frac{d\Phi}{dx}+S\Phi\right|^2+\left(V+\frac{dS}{dx}-S^2\right)|\Phi|^2=\omega^2|\Phi|^2$$

For continuous S

$$-\left[\bar{\Phi}\frac{d\Phi}{dx}+S|\Phi|^2\right]_{-\infty}^{\infty}+\int dx\left[\left|\frac{d\Phi}{dx}+S\Phi\right|^2+\left(V+\frac{dS}{dx}-S^2\right)|\Phi|^2\right]=\omega^2\int dx\,|\Phi|^2$$

We can say $\omega^2\geq 0$ if $V+\frac{dS}{dx}-S^2\geq 0$

In general, it is hard to find an appropriate S analytically

In that case, numerical approach (e.g. solving PDE) was used so far

Today’s talk

We propose a simple method for finding an appropriate S-deformation

Also, extend this method to coupled systems

Very easy method

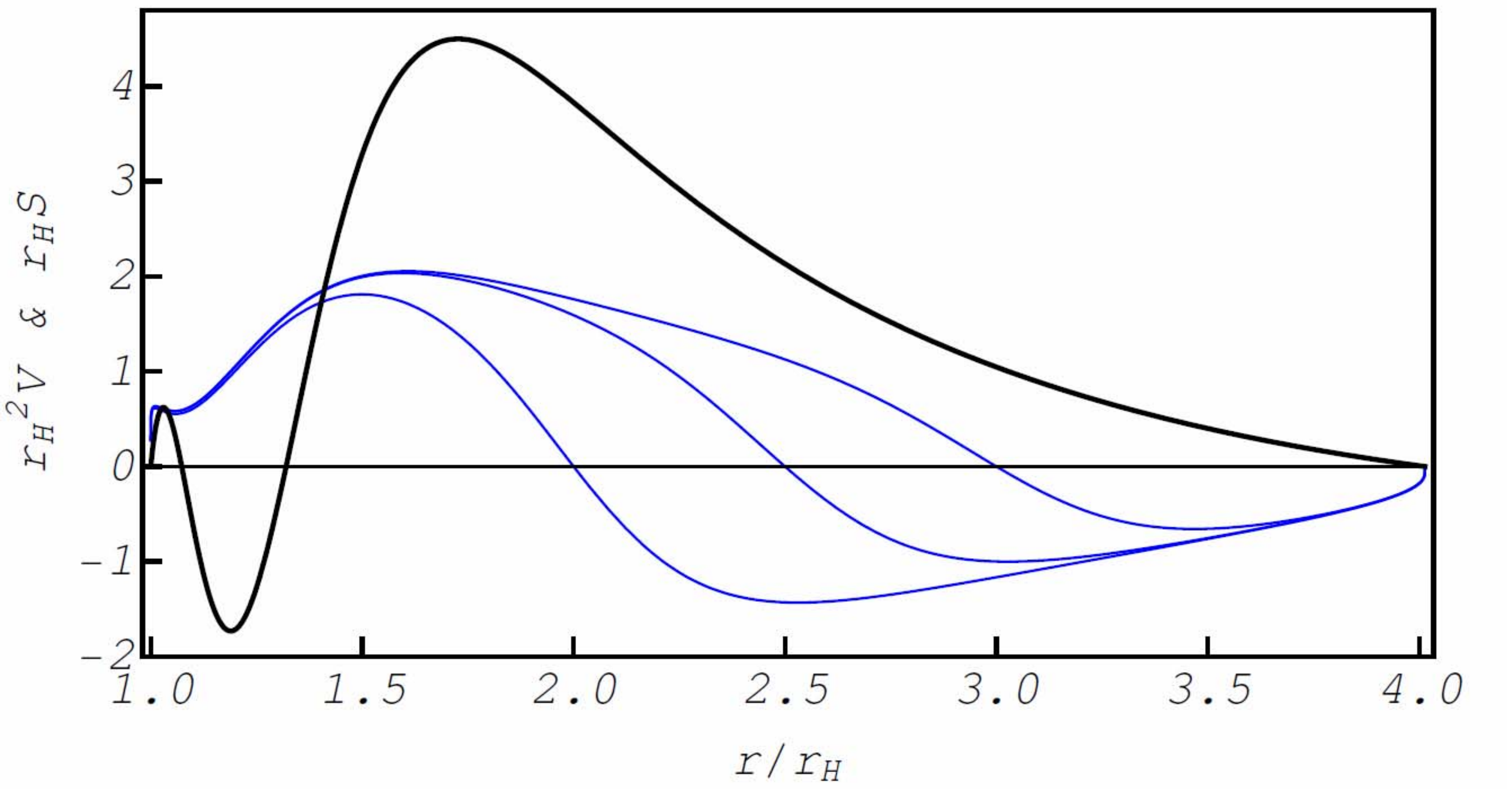
[Kimura 2017]
[Kimura & Tanaka2018]

Just solve $V + \frac{dS}{dx} - S^2 = 0$ numerically

The existence of global regular solution is non-trivial

Regular S usually can be obtained from the initial condition $S = 0$ at $V > 0$ region

10 Dim Schwarzschild-dS BH



We can find regular S without fine-tuning

Extension to multiple degrees of freedom

If there exist two or more physical degrees of freedom, and they are coupled, master Eqs sometimes become

$$\left[-\frac{d^2}{dx^2}+V\right]\Phi=\omega^2\Phi$$

V : $n \times n$ Hermitian matrix

Φ : n components vector

We assume the coupling term $\mathcal{L} \sim \Phi^\dagger V \Phi$

For any $n \times n$ Hermitian S,

$$-\left[\boldsymbol{\Phi}^\dagger \frac{d\boldsymbol{\Phi}}{dx} + \boldsymbol{\Phi}^\dagger \boldsymbol{S} \boldsymbol{\Phi}\right]_{-\infty}^{\infty} + \int dx \left[\left|\frac{d\boldsymbol{\Phi}}{dx} + \boldsymbol{S}\boldsymbol{\Phi}\right|^2 + \boldsymbol{\Phi}^\dagger \underbrace{\left(\boldsymbol{V} + \frac{d\boldsymbol{S}}{dx} - \boldsymbol{S}^2\right)}_{= \tilde{\boldsymbol{V}}} \boldsymbol{\Phi}\right] = \omega^2 \int dx \, |\boldsymbol{\Phi}|^2$$

If $\tilde{\boldsymbol{V}}$ is non-negative definite, spacetime is stable

We can still find a regular S by solving

$$\boldsymbol{V} + \frac{d\boldsymbol{S}}{dx} - \boldsymbol{S}^2 = \boldsymbol{0}$$

Schwarzschild BH in dCS

[Molina, Pani, Cardoso, Gualtieri 2010]

$$-\frac{d^2}{dx^2}\Phi_1 + V_{11}\Phi_1 + V_{12}\Phi_2 = \omega^2\Phi_1$$

$$f = 1 - \frac{2M}{r}$$

$$-\frac{d^2}{dx^2}\Phi_2 + V_{12}\Phi_1 + V_{22}\Phi_2 = \omega^2\Phi_2$$

$$f\, d/dr = d/dx$$

$$V_{11} = f\left[\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right]$$

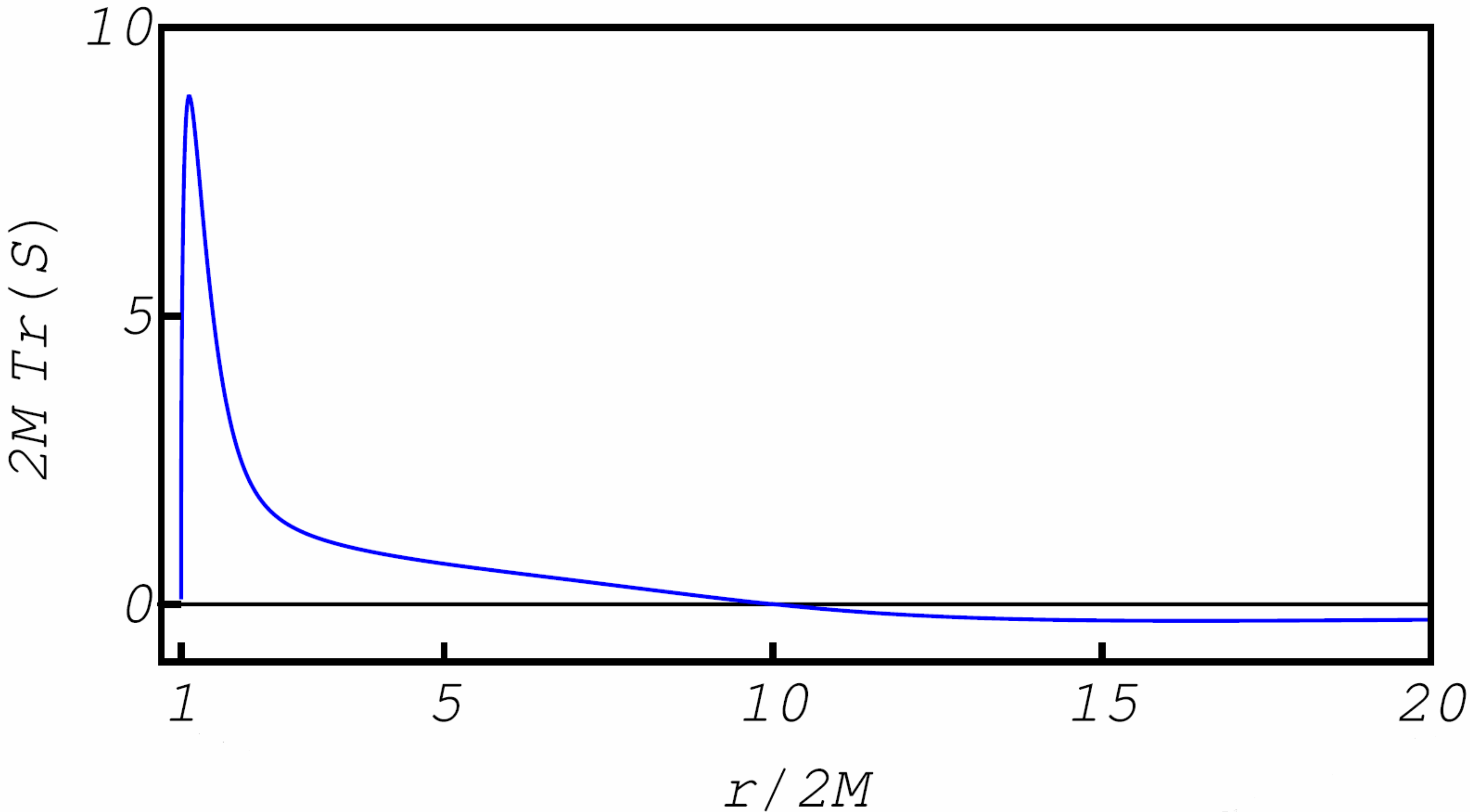
$$V_{12} = f\,\frac{24M\sqrt{\pi(\ell+2)(\ell+1)\ell(\ell-1)}}{\sqrt{\beta}r^5}$$

$$V_{22} = f\left[\frac{\ell(\ell+1)}{r^2}\left(1 + \frac{576\pi M^2}{\beta r^6}\right) + \frac{2M}{r^3}\right]$$

Schwarzschild BH in dCS

We solve $V + \frac{dS}{dx} - S^2 = 0$ numerically

with the initial condition
S = 0 at a large distance



$$\ell = 2, \beta M^4 = 1/10$$

$$V + \frac{dS}{dx} - S^2 = 0$$

(S is bounded if Tr(S) is bounded)

Comments

Based on a mathematical arguments,
we also obtained a suggestion s.t.
a regular S exists if the spacetime is stable

(see MK & T.Tanaka, arXiv:1809.00795)

If V > 0 in asymptotic region,
$S = 0$ at large x is a candidate for
an appropriate initial condition

Summary

We proposed a simple method for finding S-deformation by solving $V + \frac{dS}{dx} - S^2 = 0$

This is a good test for stability of BH

If stable, this method should work

We can guess the threshold of the parameter where unstable mode appears

Relation with Schrödinger Eq.

$$V + \frac{dS}{dx} - S^2 = 0$$ is the Riccati equation

$$\frac{1}{\phi}\frac{d\phi}{dx} := -S \quad \to \quad -\frac{d^2\phi}{dx^2} + V\phi = 0$$

Schrödinger Eq. with zero energy

A solution which does not have any zero corresponds to a regular S

Nodal theorem

A theorem in the Sturm−Liouville theory

$$\left[-\frac{d^2}{dx^2}+V\right]\Phi = E\Phi$$

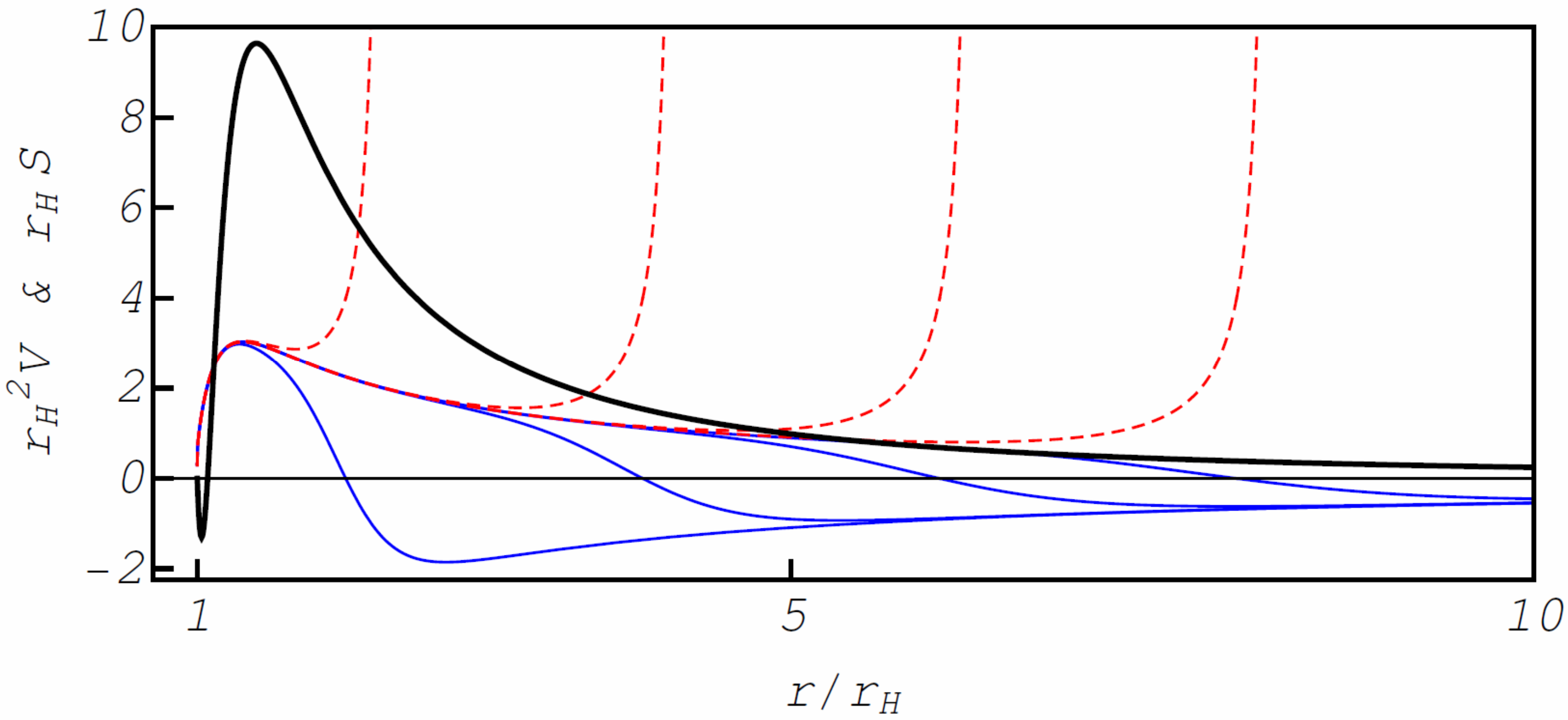
If we solve the Schrödinger Eq. with the boundary condition $\Phi = 0, d\Phi/dx = 1$ at a sufficiently large distance, the number of zeros coincides with the number of the negative energy bound states.

There should exist a regular S for stable spacetime

Under some assumption, we can show that S constructed from a sol. with decaying boundary condition is regular if the spacetime is stable.

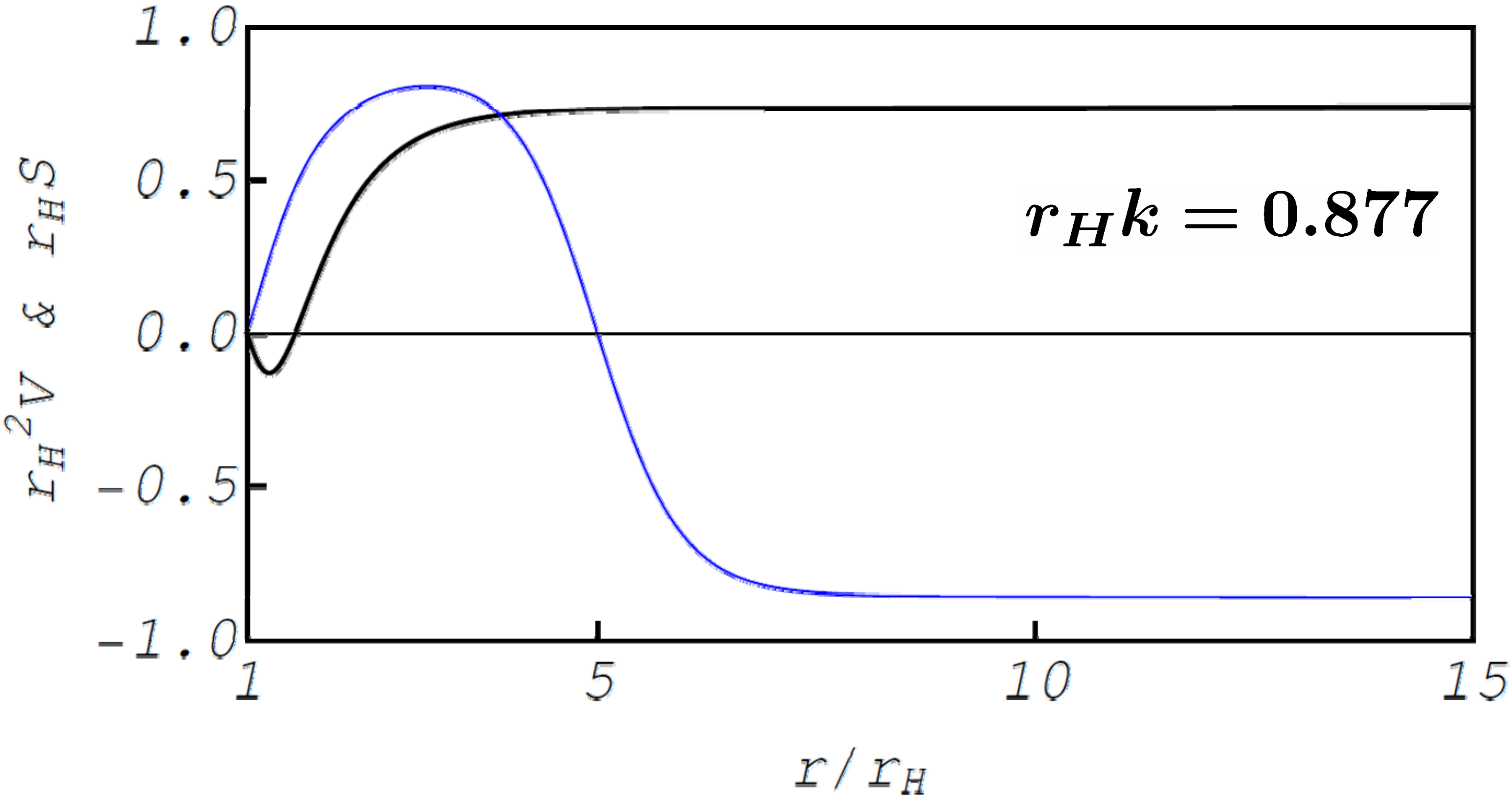
Proposition. *There exists a regular S-deformation for stable spacetimes*

10 Dim Schwarzschild BH



We can find regular S without fine-tuning

5D Black string



(If $r_H k < r_H k_{\rm cr} \simeq 0.876$ there exists an unstable mode [Gregory and Laflamme, 1993])

general regular S

Φ_L : decaying at $x \to -\infty$

Φ_R : decaying at $x \to \infty$

$$\left[-\frac{d^2}{dx^2} + V\right]\Phi = 0$$

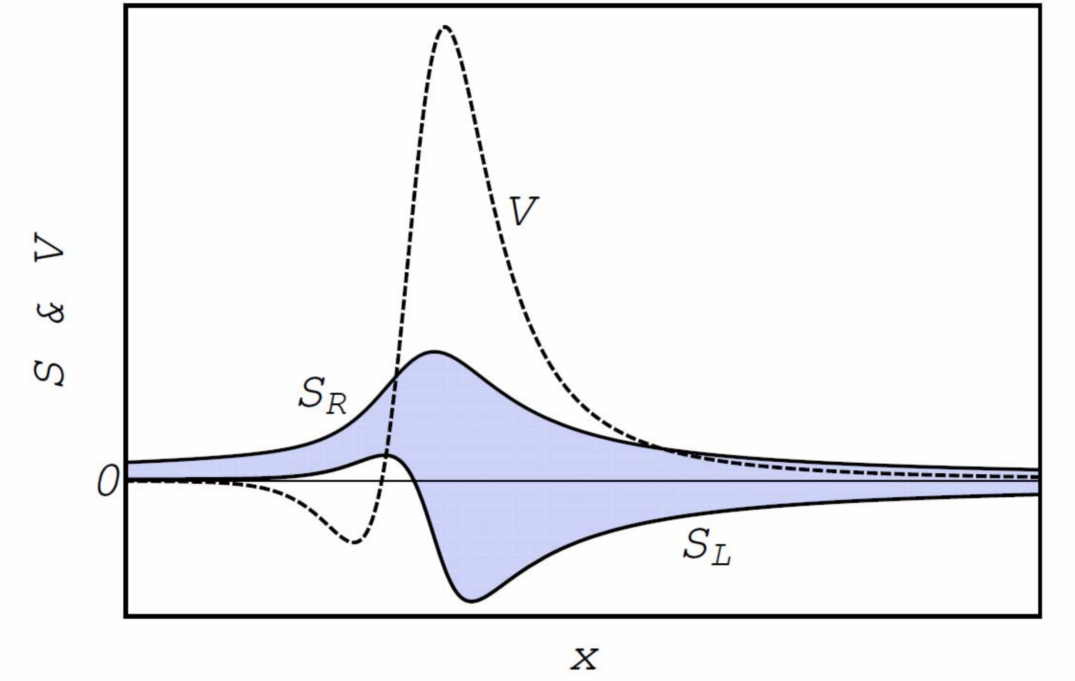
$$S_L := -\frac{1}{\Phi_L}\frac{d\Phi_L}{dx} \qquad S_R := -\frac{1}{\Phi_R}\frac{d\Phi_R}{dx}$$

General regular S is given by

$$S = -\frac{1}{\Phi}\frac{d\Phi}{dx}$$ with $$\Phi = c_L\Phi_L + c_R\Phi_R$$

$$(c_L c_R \geq 0, c_L^2 + c_R^2 \neq 0)$$

This satisfies $S_L \leq S \leq S_R$



Eq. for S

$$V + \frac{dS}{dx} - S^2 = 0$$

Shaded region corresponds to boundary conditions for regular S

If V > 0 in asymptotic region,

$S_L < 0 < S_R$ there

$S = 0$ at large x is an appropriate BC

Merit of S-deformation method

- We do not need to care about boundary condition at infinity very much, we can solve equation from finite point
- Any fine–tuning is not needed
- It is clear that the existence of regular S is the sufficient condition for stability (proof of nodal theorem is very difficult)
- Easy to show the non–existence of zero mode (by showing two different S)

Comment on $\tilde{V} \geq 0$

$$\tilde{V} = V + \frac{dS}{dx} - S^2$$

If two different regular S exists for $\tilde{V} = 0$

$$V + \frac{dS_1}{dx} - S_1^2 = 0 \qquad V + \frac{dS_2}{dx} - S_2^2 = 0$$

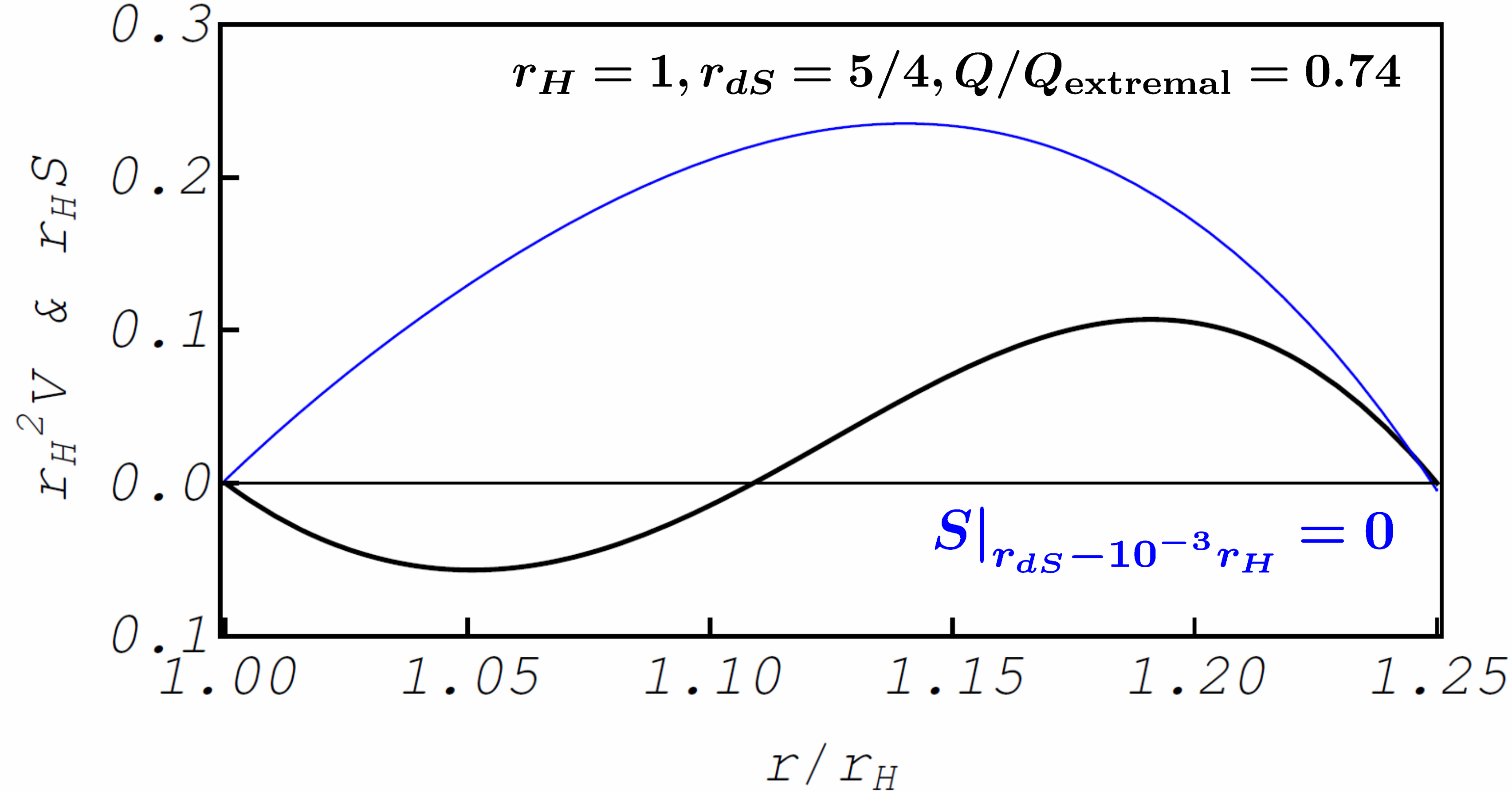
we can construct S with a non-negative $\tilde{V}$

$$S_3 := (S_1 + S_2)/2$$

$$\tilde{V} = V + \frac{dS_3}{dx} - S_3^2 = \frac{(S_1 - S_2)^2}{4} \geq 0$$

(This can be generalized a bit more...)

10 Dim RN-dS BH



If $Q/Q_{\text{extremal}} > q_{\text{cr}} \simeq 0.75$, there exists an unstable mode [Konoplya and Zhidenko, 2008]

Local existence $V + \frac{dS}{dx} - S^2 = 0$

Uniqueness theorem for ODE says there exists a solution locally

While $\frac{dS}{dx} - S^2$ is finite, S might be divergent

Global existence is not trivial

Supersymmetric quantum mechanics

From $V + \frac{dS}{dx} - S^2 = 0$

$$\left(-\frac{d}{dx} + S\right)\left(\frac{d}{dx} + S\right)\Phi = \omega^2 \Phi$$

Supersymmetric quantum mechanics system